

Mesoscopic central limit theorem for general β -ensembles

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Abstract

We prove that the linear statistics of eigenvalues of β -log gasses satisfying the one-cut and off-critical assumption with a potential $V \in C^6(\mathbb{R})$ satisfy a central limit theorem at all mesoscopic scales $\alpha \in (0; 1)$. We prove this for compactly supported test functions $f \in C^5(\mathbb{R})$ using loop equations at all orders along with rigidity estimates.

1 Introduction

We consider a system of N particles on the real line distributed according to a density proportional to

$$\prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta e^{-N \sum V(\lambda_i)} \prod d\lambda_i ,$$

where V is a continuous potential and $\beta > 0$. This system is called the β -log gas, or general β -ensemble and for classical values of $\beta \in \{1, 2, 4\}$, this distribution corresponds to the joint law of the eigenvalues of symmetric, hermitian or quaternionic random matrices with density proportional to $e^{-N \text{Tr } V(M)} dM$ where N is the size of the random matrix M .

Recently, great progress has been made to understand the behaviour of β -log gasses. At the microscopic scale, the eigenvalues exhibit a universal behaviour (see [3], [7], [6], [2]) and the local statics of the eigenvalues are described by the $Sine_\beta$ process in the bulk and the Stochastic Airy Operator at the edge (see [19] and [17] for definitions). At the macroscopic level, the eigenvalues satisfy a central limit theorem and the re-centered linear statistics of the eigenvalues converge towards a Gaussian random variable. This was first proved in [14] for polynomial potentials satisfying the one-cut assumption. In [5], the authors derived a full expansion of the free energy in the one-cut regime from which they deduce the central limit theorem for analytic potentials. The multi-cut

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regime is more complicated and in this setting, the central limit theorem does not hold anymore for all test functions (see [4], [18]). In this article, we consider the scale between microscopic and macroscopic called the mesoscopic regime. Specifically, we study the linear fluctuations of the eigenvalues of general β -ensembles at the mesoscopic scale; we prove that for $\alpha \in (0; 1)$ fixed, f a smooth function (whose regularity and decay at infinity will be specified later), and E a fixed energy level

$$\sum_{i=1}^N f(N^\alpha(\lambda_i - E)) - N \int f(N^\alpha(x - E)) d\mu_V(x)$$

converges towards a Gaussian random variable.

Interest in mesoscopic linear statistics has surged in recent years. Results in this field of study were obtained in a variety of settings, for Gaussian random matrices [9, 12], and for invariant ensembles [11, 15]. In many cases the results were shown at all scales $\alpha \in (0; 1)$, often with the use of distribution specific properties. In more general settings, the absence of such properties necessitates other approaches to obtain the limiting behaviour at the mesoscopic regime. For example, an early paper studying mesoscopic statistics for Wigner Matrices was [10], here the regime studied was $\alpha \in (0; \frac{1}{8})$, later using improved local law results this was pushed to $\alpha \in (0; \frac{1}{3})$ [16], and recent work has pushed this to all scales [13].

Extending these results to general β -ensembles is a natural step. We also prove convergence at all mesoscopic scales. The proof of the main Theorem relies on the analysis of the loop equations from which we can deduce a recurrence relationship between moments, and the rigidity results from [7], [6] to control the linear statistics. Similar results have been obtained before in [8, Theorem 5.4]. There, the authors showed the mesoscopic CLT in the case of a quadratic potential, for small α (see Remark 5.5).

In Section 1, we introduce the model and recall some background results and Section 2 will be dedicated to the proof of 1.4.

1.1 Definitions and Background

We consider the general β -matrix model. For a potential $V : \mathbb{R} \rightarrow \mathbb{R}$ and $\beta > 0$, we denote the measure on \mathbb{R}^N

$$\mathbb{P}_V^N(d\lambda_1, \dots, d\lambda_N) := \frac{1}{Z_V^N} \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta e^{-N \sum V(\lambda_i)} \prod d\lambda_i, \quad (1.1)$$

with

$$Z_V^N = \int \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta e^{-N \sum V(\lambda_i)} \prod d\lambda_i.$$

It is well known that under \mathbb{P}_V^N the empirical measure of the eigenvalues converge towards an equilibrium measure:

Theorem 1.1. *Assume that $V : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and that*

$$\liminf_{x \rightarrow \infty} \frac{V(x)}{\beta \log |x|} > 1.$$

Then the energy defined by

$$E(\mu) = \iint \left(\frac{V(x_1) + V(x_2)}{2} - \frac{\beta}{2} \log |x_1 - x_2| \right) d\mu(x_1) d\mu(x_2) \quad (1.2)$$

has a unique global minimum on the space $\mathcal{M}_1(\mathbb{R})$ of probability measures on \mathbb{R} .

Moreover, under \mathbb{P}_V^N the normalized empirical measure $L_N = N^{-1} \sum_{i=1}^N \delta_{\lambda_i}$ converges almost surely and in expectation towards the unique probability measure μ_V which minimizes the energy.

Furthermore, μ_V has compact support A and is uniquely determined by the existence of a constant C such that:

$$\beta \int \log |x - y| d\mu_V(y) - V(x) \leq C ,$$

with equality almost everywhere on the support. The support of μ_V is a union of intervals $A = \bigcup_{0 \leq h \leq g} [\alpha_{h,-}; \alpha_{h,+}]$ with $\alpha_{h,-} < \alpha_{h,+}$ and if V is smooth on a neighbourhood of A ,

$$\frac{d\mu_V}{dx} = S(x) \prod_{h=0}^g \sqrt{|x - \alpha_{h,-}| |x - \alpha_{h,+}|} ,$$

with S smooth on a neighbourhood of A .

1.2 Results

Hypothesis 1.2. For what proceeds, we assume the following

- V is continuous and goes to infinity faster than $\beta \log|x|$.
- The support of μ_V is a connected interval $A = [a; b]$ and

$$\frac{d\mu_V}{dx} = \rho_V(x) = S(x) \sqrt{(b-x)(x-a)} \quad \text{with } S > 0 \text{ on } [a; b].$$

- The function $V(\cdot) - \beta \int \log |\cdot - y| d\mu_V(y)$ achieves its minimum on the support only.

Remark 1.3. The second and third assumptions are typically known as the one-cut and off-criticality assumptions. In the case where the support of the equilibrium measure is no longer connected, the macroscopic central limit theorem does not hold anymore in generality (see [4] , [18]). Whether the theorem holds for critical potentials is still an open question.

Theorem 1.4. Let $0 < \alpha < 1$, E a point in the bulk $(a; b)$, $V \in C^6(\mathbb{R})$ and $f \in C^5(\mathbb{R})$ with compact support. Then, under \mathbb{P}_V^N

$$\sum_{i=1}^N f(N^\alpha(\lambda_i - E)) - N \int f(N^\alpha(x - E)) d\mu_V(x) \xrightarrow{\mathcal{M}} \mathcal{N}(0, \sigma_f^2) ,$$

where the convergence holds in moments (and thus, in distribution), and

$$\sigma_f^2 = \frac{1}{2\beta\pi^2} \iint \left(\frac{f(x) - f(y)}{x - y} \right)^2 dx dy .$$

Note that, as in the macroscopic central limit theorem, the variance is universal in the potential with a multiplicative factor proportional to β . Interestingly and in contrast with the macroscopic scale, the limit is always centered.

The proof relies on an explicit computation of the moments of the linear statistics. We will use two tools: optimal rigidity for the eigenvalues of beta-ensembles to provide a bound on the linear statistics (as in [7], [6]) and the loop equations at all orders to derive a recurrence relationship between the moments.

Acknowledgements

The authors would like to thank Alice Guionnet for the fruitful discussions and the very helpful comments.

2 Proof of 1.4

For what follows, set

$$L_N = \frac{1}{N} \sum_i \delta_{\lambda_i}, \quad M_N = \sum_{i=1}^N \delta_{\lambda_i} - N\mu_V.$$

and for a measure ν and an integrable function h set

$$\nu(h) = \int h d\nu \quad \text{and} \quad \tilde{\nu}(h) = \int h d\nu - \mathbb{E}_V^N \left(\int h d\nu \right),$$

when ν is random and where \mathbb{E}_V^N is expectation with respect to \mathbb{P}_V^N . Further f will be by any function as in 1.4, and

$$\tilde{f}(x) := f(N^\alpha(x - E)).$$

Finally, for any function $g \in C^p(\mathbb{R})$, let

$$\|g\|_{C^p(\mathbb{R})} := \sum_{l=0}^p \sup_{x \in \mathbb{R}} |g^{(l)}(x)|,$$

when it exists.

2.1 Loop Equations

To prove the convergence, we use the loop equations at all orders. Loop equations have been used previously to derive recurrence relationships between correlators and derive a full expansion of the free energy for β -ensembles in [18], [4], and [5] (from which the authors also derive a macroscopic central limit theorem). The first loop equation was used to prove the central limit theorem at the macroscopic scale in [14] and used subsequently in [8]. Here, rather than using the first loop equation to control the Stieltjes transform as in [14] and [8], we rely on the analysis of the loop equations at all orders to compute directly the moments.

Proposition 2.1. *Let h, h_1, h_2, \dots be a sequence of functions in $C^1(\mathbb{R})$. Define*

$$F_1^N(h) = \frac{\beta}{2} \iint \frac{h(x) - h(y)}{x - y} dL_N(x) dL_N(y) - L_N(hV') + \frac{1}{N} \left(1 - \frac{\beta}{2}\right) L_N(h') \quad (2.1)$$

and for all $k \geq 1$

$$F_{k+1}^N(h, h_1, \dots, h_k) = F_k^N(h, h_1, \dots, h_{k-1}) \tilde{M}_N(h_k) + \left(\prod_{l=1}^{k-1} \tilde{M}_N(h_l) \right) L_N(hh'_k) \quad (2.2)$$

where the product is equal to 1 when $k = 1$. Then we have for all $k \geq 1$

$$\mathbb{E}_V^N(F_k^N(h, h_1, \dots, h_{k-1})) = 0. \quad (2.3)$$

Proof. The first loop equation is derived by integration by parts. We derive the loop equation at order $k + 1$ from the one at order k by replacing V by $V + \delta h_k$ and differentiating at $\delta = 0$. \square

It will be easier to compute recursively the moments by re-centering the first loop equation. To that end, define the operator Ξ acting on smooth functions $h : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\Xi h(x) = \beta \int \frac{h(x) - h(y)}{x - y} d\mu_V(y) - V'(x)h(x).$$

We then use equilibrium relations to recenter L_N by μ_V . Consider for δ in a neighbourhood of 0, $\mu_{V,\delta} = (x + \delta h(x))\sharp \mu_V$, where for a map T and measure μ , $T\sharp \mu$ refers to the push-forward measure of μ by T . Then by (1.2) we have $E(\mu_{V,\delta}) \geq E(\mu_V)$. By differentiating at $\delta = 0$ we obtain

$$\frac{\beta}{2} \iint \frac{h(x) - h(y)}{x - y} d\mu_V(x) d\mu_V(y) = \int V'(x) f(x) d\mu_V(x), \quad (2.4)$$

and thus

$$\begin{aligned} \frac{\beta}{2} \iint \frac{h(x) - h(y)}{x - y} dL_N(x) dL_N(y) - L_N(hV') = \\ \frac{1}{N} M_N(\Xi h) + \frac{\beta}{2N^2} \iint \frac{h(x) - h(y)}{x - y} dM_N(x) dM_N(y). \end{aligned}$$

Consequently, we can write

$$F_1^N(h) = M_N(\Xi h) + \left(1 - \frac{\beta}{2}\right) L_N(h') + \frac{1}{N} \left[\frac{\beta}{2} \iint \frac{h(x) - h(y)}{x - y} dM_N(x) dM_N(y) \right]. \quad (2.5)$$

One of the key features of the operator Ξ is that it is invertible (modulo constants) in the space of smooth functions. More precisely, we have the following Lemma (see Lemma 3.2 of [3] for the proof):

Lemma 2.2. *Inversion of Ξ*

Assume that $V \in C^p(\mathbb{R})$ and satisfies Hypothesis 1.2. Let $[a; b]$ denote the support of μ_V and set

$$\frac{d\mu_V}{dx} = S(x)\sqrt{(b-x)(x-a)} = S(x)\sigma(x),$$

where $S > 0$ on $[a; b]$.

Then for any $k \in C^r(\mathbb{R})$ there exists a unique constant c_k and $h \in C^{(r-2) \wedge (p-3)}(\mathbb{R})$ such that

$$\Xi(h) = k + c_k.$$

Moreover the inverse is given by the following formulas:

- $\forall x \in \text{supp}(\mu_V)$

$$h(x) = -\frac{1}{\beta\pi^2 S(x)} \left(\int_a^b \frac{k(y) - k(x)}{\sigma(y)(y-x)} dy \right) \quad (2.6)$$

- $\forall x \notin \text{supp}(\mu_V)$

$$h(x) = \frac{\beta \int \frac{h(y)}{x-y} d\mu_V(y) - k(x) - c_k}{\beta \int \frac{1}{x-y} d\mu_V(y) - V'(x)}. \quad (2.7)$$

Note that the definition (2.7) is proper since h has been defined on the support.

We shall denote this inverse by $\Xi^{-1}k$.

Remark 2.3. For f and V as in 1.4, $p = 6$ and $r = 5$ so $\Xi^{-1}\tilde{f} \in C^3(\mathbb{R})$.

In order to bound the linear statistics we use the following lemma to bound $\Xi^{-1}(\tilde{f})$ and its derivatives.

Lemma 2.4. Let $\text{supp } f \subset [-M, M]$ for some constant $M > 0$. For each $p \in \{1, 2, 3\}$, there is a constant $C > 0$ such that

$$\left\| \Xi^{-1}(\tilde{f}) \right\|_{C^p(\mathbb{R})} \leq CN^{p\alpha} \log N, \quad (2.8)$$

Moreover, there is a constant C such that whenever $N^\alpha|x - E| \geq M + 1$

$$\left| \Xi^{-1}(\tilde{f})^{(p)}(x) \right| \leq \frac{C}{N^\alpha((x - E)^{p+1} \wedge 1)}, \quad (2.9)$$

Proof. We start with (2.8). Using (2.7), we see that $\Xi^{-1}(\tilde{f})$ and its derivatives are clearly uniformly bounded outside $\text{supp } \mu_V$. For $x \in \text{supp } \mu_V$ we use

$$\Xi^{-1}(\tilde{f})(x) = -\frac{N^\alpha}{\beta\pi^2 S(x)} \int_a^b \frac{1}{\sigma(y)} \int_0^1 f'(N^\alpha t(x - E) + N^\alpha(1-t)(y - E)) dt dy$$

so that

$$\begin{aligned} \Xi^{-1}(\tilde{f})^{(p)}(x) = & -\frac{1}{\beta\pi^2} \sum_{l=0}^p \left\{ \binom{p}{l} \left(\frac{1}{S} \right)^{(p-l)}(x) \right. \\ & \times \left. \int_a^b \frac{N^{(l+1)\alpha}}{\sigma(y)} \int_0^1 t^l f^{(l+1)}(N^\alpha t(x - E) + N^\alpha(1-t)(y - E)) dt dy \right\}. \end{aligned}$$

Let $A(x) = \{(t, y) \in [0; 1] \times [a; b] , N^\alpha |t(x - E) + (1 - t)(y - E)| \leq M\}$. We have

$$\int_0^1 \mathbb{1}_{A(x)}(t, y) dt \leq \frac{2M}{N^\alpha |x - y|} \wedge 1 \quad (2.10)$$

and thus

$$\int_a^b \frac{N^{(l+1)\alpha}}{\sigma(y)} \int_0^1 |f^{(l+1)}(N^\alpha t(x - E) + N^\alpha(1 - t)(y - E))| dt dy \leq C \log N N^{l\alpha},$$

and this proves (2.8).

We now proceed with the proof of (2.9). First, let $x \in \text{supp } \mu_V$ such that $N^\alpha |x - E| \geq M + 1$. The inversion formula (2.6) writes

$$\begin{aligned} \Xi^{-1}(\tilde{f})(x) &= -\frac{1}{\beta\pi^2 S(x)} \int_a^b \frac{f(N^\alpha(y - E))}{\sigma(y)(y - x)} dy \\ &= -\frac{1}{\beta\pi^2 S(x)} \int_{-M}^M \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})(u - N^\alpha(x - E))} du. \end{aligned} \quad (2.11)$$

By differentiating this formula, we obtain (2.9) for $x \in \text{supp } \mu_V$. The result for $x \notin \text{supp } \mu_V$ is obtained similarly using (2.7). \square

2.2 Control of the linear statistics

We now make use of the strong rigidity estimates proved in [6] (Theorem 2.4) to control the linear statistics. We recall the result here

Theorem 2.5. *Let γ_i the quantile defined by*

$$\int_a^{\gamma_i} d\mu_V(x) = \frac{i}{N}. \quad (2.12)$$

Then, under Hypothesis 1.2 and for all $\xi > 0$ there exists constants $c > 0$ such that for N large enough

$$\mathbb{P}_V^N(|\lambda_i - \gamma_i| \geq N^{-2/3+\xi} \hat{i}^{-1/3}) \leq e^{-N^c},$$

where $\hat{i} = i \wedge (N + 1 - i)$.

We will use the following lemma quite heavily in what proceeds.

Lemma 2.6. *Let γ_i and \hat{i} be as in Theorem 2.5, let $t \in [0; 1]$, and let λ_i , $i \in \llbracket 1, N \rrbracket$, be a configuration of points such that $|\lambda_i - \gamma_i| \leq N^{-2/3+\xi} \hat{i}^{-1/3}$ for $0 < \xi < (1 - \alpha) \wedge \frac{2}{3}$, and let $M > 1$ be a constant. Define the pairwise disjoint sets:*

$$J_1 := \{i \in \llbracket 1; N \rrbracket, |N^\alpha(\gamma_i - E)| \leq 2M\}, \quad (2.13)$$

$$J_2 := \left\{ i \in J_1^c, |(\gamma_i - E)| \leq \frac{1}{2}(E - a) \wedge (b - E) \right\}, \quad (2.14)$$

$$J_3 := J_1^c \cap J_2^c. \quad (2.15)$$

The following statements hold:

(a) For all $i \in J_1 \cup J_2$, $\hat{i} \geq CN$, for some $C > 0$ that depend only on μ_V in a neighborhood of E , also for all such i , $|\gamma_i - \gamma_{i+1}| \leq \frac{C}{N}$ for a constant $C > 0$ depending only on μ_V in a neighborhood of E .

(b) Uniformly in all $i \in J_1^c = J_2 \cup J_3$ and all $t \in [0; 1]$,

$$|N^\alpha t(\lambda_i - \gamma_i) + N^\alpha(\gamma_i - E)| > M + 1, \quad (2.16)$$

for large enough N . Furthermore, the statement holds true uniformly in $x \in [\gamma_i, \gamma_{i+1}]$ when we substitute γ_i by x .

(c) The cardinality of J_1 is of order $CN^{1-\alpha}$, where again, $C > 0$ depends only on μ_V in a neighborhood of E .

Proof. The first part of statement (a) holds by the observation that for $i \in J_1 \cup J_2$, γ_i is in the bulk, so

$$0 < c \leq \int_a^{\gamma_i} d\mu_V(x) = \frac{i}{N} \leq C < 1$$

for constants $C, c > 0$ depending only on μ_V . For the second part of statement (a), the density of μ_V is bounded below uniformly in $i \in J_1 \cup J_2$, so

$$c|\gamma_i - \gamma_{i+1}| \leq \int_{\gamma_i}^{\gamma_{i+1}} d\mu_V(x) = \frac{1}{N}.$$

Statement (b) can be seen as follows: consider $i \in J_2$, on this set $\hat{i} \geq CN$ by (a), so uniformly in such i , $N^\alpha|\lambda_i - \gamma_i| \leq CN^{\alpha-1+\xi}$, which goes to zero, while $N^\alpha|\gamma_i - E| > 2M$. On the other hand, for $i \in J_3$, we have $N^\alpha|\gamma_i - E| > \frac{1}{2}N^\alpha(E - a) \wedge (b - E)$, which goes to infinity faster than $N^\alpha|\lambda_i - \gamma_i| \leq N^{\alpha-\frac{2}{3}+\xi}$, by our choice of ξ . When we substitute γ_i by x , the same argument holds because $N^\alpha|x - \gamma_i| \leq N^\alpha|\gamma_i - \gamma_{i+1}|$, which is of order $N^{\alpha-1}$ on J_2 (as we showed in statement (a)) and of order $CN^{\alpha-\frac{2}{3}}$ on J_3 .

Statement (c) follows by the observation that on the set $x \in [a, b]$ such that $|x - E| \leq \frac{2M}{N^\alpha}$ the density of μ_V is bounded uniformly above and below, so

$$\frac{c}{N^\alpha} \leq \int_{|x-E| \leq \frac{2M}{N^\alpha}} d\mu_V(x) = \sum_{i \in J_1} \int_{\gamma_i}^{\gamma_{i+1}} d\mu_V(x) + O\left(\frac{1}{N}\right) \leq \frac{C}{N^\alpha},$$

giving the required result. \square

The rigidity of eigenvalues, 2.5, along with the previous Lemma leads to the following estimates

Lemma 2.7. For all $0 < \xi < (1 - \alpha) \wedge \frac{2}{3}$ there exists constants $C, c > 0$ such that for N large enough we have the concentration bounds

$$\mathbb{P}_V^N(|M_N(\tilde{f})| \geq CN^\xi \|f\|_{C^1(\mathbb{R})}) \leq e^{-N^c}, \quad (2.17)$$

$$\mathbb{P}_V^N(|M_N(\Xi^{-1}(\tilde{f})')| \geq CN^{\alpha+\xi} \|f\|_{C^1(\mathbb{R})}) \leq e^{-N^c}, \quad (2.18)$$

$$\mathbb{P}_V^N(|M_N(\Xi^{-1}(\tilde{f})\tilde{f}')| \geq CN^{\alpha+\xi} \|f\|_{C^1(\mathbb{R})}) \leq e^{-N^c}. \quad (2.19)$$

Proof. Let $M > 1$ such that $\text{supp } f \subset [-M, M]$ and fix $0 < \xi < (1 - \alpha) \wedge \frac{2}{3}$. For the remainder of the proof, we may assume that we are on the event $\Omega := \{\forall i, |\lambda_i - \gamma_i| \leq N^{-2/3+\xi} \hat{i}^{-1/3}\}$. This follows from the fact that, for example,

$$\mathbb{P}_N^V(|M_N(\tilde{f})| \geq CN^\xi \|f\|_{C^1(\mathbb{R})}) \leq \mathbb{P}_N^V\left(\left\{|M_N(\tilde{f})| \geq CN^\xi \|f\|_{C^1(\mathbb{R})}\right\} \cap \Omega\right) + \mathbb{P}_N^V(\Omega^c),$$

and by 2.5, we may bound $\mathbb{P}_N^V(\Omega^c)$ by e^{-N^c} for some constant $c > 0$, and N large enough. On Ω , the λ_i satisfy the conditions of 2.6, we will utilize the sets J_1 , J_2 , and J_3 as defined there.

We begin by controlling (2.17). We have that

$$\begin{aligned} |M_N(\tilde{f})| &= \left| \sum_{i=1}^N f(N^\alpha(\lambda_i - E)) - N\mu_V(\tilde{f}) \right| \\ &\leq \left| \sum_{i=1}^N f(N^\alpha(\lambda_i - E)) - \sum_{i=1}^N f(N^\alpha(\gamma_i - E)) \right| + \left| \sum_{i=1}^N f(N^\alpha(\gamma_i - E)) - N\mu_V(\tilde{f}) \right|, \end{aligned} \quad (2.20)$$

the first term in (2.20) may be bounded (on Ω) by

$$\begin{aligned} &\left| \sum_{i=1}^N f(N^\alpha(\lambda_i - E)) - \sum_{i=1}^N f(N^\alpha(\gamma_i - E)) \right| \\ &= \left| \sum_{i=1}^N N^\alpha(\lambda_i - \gamma_i) \int_0^1 f'(tN^\alpha(\lambda_i - \gamma_i) + N^\alpha(\gamma_i - E)) dt \right| \\ &\leq \sum_{i=1}^N N^{\alpha-2/3+\xi} \hat{i}^{-1/3} \int_0^1 |f'(tN^\alpha(\lambda_i - \gamma_i) + N^\alpha(\gamma_i - E))| dt, \end{aligned}$$

By 2.6 (b), for N large enough, we have

$$\begin{aligned} &\int_0^1 \sum_{i=1}^N N^{\alpha-2/3+\xi} \hat{i}^{-1/3} |f'(tN^\alpha(\lambda_i - \gamma_i) + N^\alpha(\gamma_i - E))| dt \\ &= \int_0^1 \sum_{i \in J_1} N^{\alpha-2/3+\xi} \hat{i}^{-1/3} |f'(tN^\alpha(\lambda_i - \gamma_i) + N^\alpha(\gamma_i - E))| dt \\ &\leq \sum_{i \in J_1} N^{\alpha-1+\xi} \|f\|_{C^1(\mathbb{R})} \leq CN^\xi \|f\|_{C^1(\mathbb{R})}, \end{aligned} \quad (2.21)$$

where, in the third line we used 2.6 (a) and (c) in order. Thus

$$\left| \sum_{i=1}^N f(N^\alpha(\lambda_i - E)) - \sum_{i=1}^N f(N^\alpha(\gamma_i - E)) \right| \leq CN^\xi \|f\|_{C^1(\mathbb{R})}.$$

For the second term in (2.20),

$$\begin{aligned}
& \left| \sum_{i=1}^N f(N^\alpha(\gamma_i - E)) - N \int_a^b f(N^\alpha(x - E)) d\mu_V(x) \right| \\
& \leq N \sum_{i \in J_1} \int_{\gamma_i}^{\gamma_{i+1}} |f(N^\alpha(\gamma_i - E)) - f(N^\alpha(x - E))| d\mu_V(x) \\
& \leq N^{1+\alpha} \|f\|_{C^1(\mathbb{R})} |J_1| \sup_{i \in J_1} (\gamma_{i+1} - \gamma_i) \int_{\gamma_i}^{\gamma_{i+1}} d\mu_V(x) \leq C \|f\|_{C^1(\mathbb{R})}
\end{aligned}$$

since the spacing of the quantiles in J_1 is bounded by $\frac{C}{N}$. This proves (2.17).

We now proceed with the proof of (2.18).

$$\begin{aligned}
|M_N(\Xi^{-1}(\tilde{f})')| &= \left| \sum_{i=1}^N \left(\Xi^{-1}(\tilde{f})'(\lambda_i) - N \int_{\gamma_i}^{\gamma_{i+1}} \Xi^{-1}(\tilde{f})'(x) d\mu_V(x) \right) \right| \\
&\leq N \sum_{i=1}^N \int_{\gamma_i}^{\gamma_{i+1}} \left| \Xi^{-1}(\tilde{f})'(\lambda_i) - \Xi^{-1}(\tilde{f})'(x) \right| d\mu_V(x) \\
&\leq N \sum_{i=1}^N \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 |\lambda_i - x| \left| \Xi^{-1}(\tilde{f})^{(2)}(t(\lambda_i - x) + x) \right| dt d\mu_V(x),
\end{aligned}$$

Recall from the proof of Lemma 2.6 that uniformly in $i \in J_2$ and $x \in [\gamma_i, \gamma_{i+1}]$, $|\gamma_i - E| \geq \frac{2M}{N^\alpha}$ while $|x - \gamma_i| \leq \frac{C}{N}$; further, $|\lambda_i - x| \leq CN^{-1+\xi}$ so for N large enough we can replace $|t(\lambda_i - x) + (\gamma_i - E)|$ by $|\gamma_i - E|$ uniformly in $t \in [0, 1]$. Likewise, uniformly in $i \in J_3$ and $x \in [\gamma_i, \gamma_{i+1}]$, $|\gamma_i - E| \geq C$ while $|x - \gamma_i| \leq CN^{-\frac{2}{3}}$; further $|\lambda_i - x| \leq CN^{-1+\xi}$ so for N large enough we can replace $|t(\lambda_i - x) + (\gamma_i - E)|$ by a constant C uniformly in $t \in [0, 1]$ for what follows.

For $i \in J_2$, by the observations in the previous paragraph, along with 2.6 (b), 2.4 eq. (2.9), and 2.6 (a),

$$\begin{aligned}
& N \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 |\lambda_i - x| \left| \Xi^{-1}(\tilde{f})^{(2)}(t(\lambda_i - x) + x) \right| dt d\mu_V(x) \\
& \leq N \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 \frac{C|\lambda_i - x|}{N^\alpha(|t(\lambda_i - x) + x - E|^3 \wedge 1)} dt d\mu_V(x) \leq \sum_{i \in J_2} \frac{CN^{\xi-1-\alpha}}{(\gamma_i - E)^3},
\end{aligned}$$

The same reasoning for $i \in J_3$ yields

$$N \sum_{i \in J_3} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 |\lambda_i - x| \left| \Xi^{-1}(\tilde{f})^{(2)}(t(\lambda_i - x) + x) \right| dt d\mu_V(x) \leq \sum_{i \in J_3} CN^{\xi-\alpha-\frac{2}{3}} \hat{i}^{-\frac{1}{3}}.$$

For $i \in J_1$, by 2.4 eq. (2.8) and 2.6 (a),

$$\begin{aligned}
& N \sum_{i \in J_1} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 |\lambda_i - x| \left| \Xi^{-1}(\tilde{f})^{(2)}(t(\lambda_i - x) + x) \right| dt d\mu_V(x) \\
& \leq N \sum_{i \in J_1} \int_{\gamma_i}^{\gamma_{i+1}} CN^{2\alpha} \log N |\lambda_i - x| d\mu_V(x) \leq \sum_{i \in J_1} CN^{2\alpha+\xi-1} \log N.
\end{aligned}$$

It follows that

$$\begin{aligned} \left| M_N(\Xi^{-1}(\tilde{f})') \right| &\leq \sum_{i \in J_1} C N^{2\alpha+\xi-1} \log N + \sum_{i \in J_2} \frac{C N^{\xi-1-\alpha}}{(\gamma_i - E)^3} + \sum_{i \in J_3} C N^{\xi-\alpha-\frac{2}{3}\hat{i}^{\frac{1}{3}}} \\ &\leq C N^{\alpha+\xi} \log N + C N^{\xi+\alpha} \leq C N^{\alpha+\xi} \log N, \end{aligned}$$

where we have used $|J_1| \leq C N^{1-\alpha}$ from 2.6, and the following estimates:

$$\begin{aligned} \sum_{i \in J_2} \frac{N^{\xi-\alpha-1}}{(\gamma_i - E)^3} &\leq C N^{\xi-\alpha} \left(\int_a^{E-\frac{2M}{N^\alpha}} \frac{dx}{(x-E)^3} + \int_{E+\frac{2M}{N^\alpha}}^b \frac{dx}{(x-E)^3} \right) \leq C N^{\xi+\alpha}, \\ C N^{\xi-\alpha-\frac{2}{3}} \sum_{i \in J_3} \hat{i}^{-\frac{1}{3}} &\leq C N^{\xi-\alpha} \times \frac{1}{N} \sum_{i=1}^N \left(\frac{i}{N} \right)^{-\frac{1}{3}} \leq C N^{\xi-\alpha}. \end{aligned}$$

This proves (2.18). The bound (2.19) is obtained in a similar way and we omit the details. \square

For convenience we introduce the following notation: for a sequence of random variable $(X_N)_{N \in \mathbb{N}}$ we write $X_N = \omega(1)$ if there exists constants c, C and $\delta > 0$ such that the bound $|X_N| \leq \frac{C}{N^\delta}$ holds with probability greater than $1 - e^{-N^c}$.

Using Lemma 2.7 we prove the following bounds:

Lemma 2.8. *The following estimates hold:*

$$L_N(\Xi^{-1}(\tilde{f})') = \omega(1), \quad (2.22)$$

$$L_N(\Xi^{-1}(\tilde{f})\tilde{f}') + \sigma_f^2 = \omega(1), \quad (2.23)$$

$$\frac{1}{N} \iint \frac{\Xi^{-1}(\tilde{f})(x) - \Xi^{-1}(\tilde{f})(y)}{x-y} dM_N(x) dM_N(y) = \omega(1). \quad (2.24)$$

Proof. For both (2.22) and (2.23), we use

$$\begin{aligned} L_N(\Xi^{-1}(\tilde{f})') &= \frac{M_N(\Xi^{-1}(\tilde{f})')}{N} + \mu_V(\Xi^{-1}(\tilde{f}')), \\ L_N(\Xi^{-1}(\tilde{f})\tilde{f}') &= \frac{M_N(\Xi^{-1}(\tilde{f})\tilde{f}')}{N} + \mu_V(\Xi^{-1}(\tilde{f})\tilde{f}'), \end{aligned}$$

2.7 implies that the first term in both equations are $\omega(1)$ so (2.22) and (2.23) simplify to deterministic statements about the speed of convergence of the integrals against μ_V above.

To show (2.22), integration by parts yields:

$$\int (\Xi^{-1}\tilde{f})'(x) d\mu_V(x) = - \int_a^b (\Xi^{-1}\tilde{f})(x) (S'(x)\sigma(x) + S(x)\sigma'(x)) dx,$$

inserting the formula for $\Xi^{-1}\tilde{f}$ we obtain

$$\left| \int (\Xi^{-1}\tilde{f})'(x) d\mu_V(x) \right| \leq \frac{1}{\beta\pi^2} \int_a^b \int_a^b \left| \frac{\tilde{f}(x) - \tilde{f}(y)}{y - x} \right| \left(\left| \frac{S'(x)\sigma(x)}{S(x)\sigma(y)} \right| + \left| \frac{\sigma'(x)}{\sigma(y)} \right| \right) dx dy .$$

Recall that S is bounded below on $[a, b]$, S' is bounded above on $[a, b]$, further, up to a constant, $\frac{\sigma'(x)}{\sigma(y)}$ can be bounded above by $(\sigma(x)\sigma(y))^{-1}$. We define the sets

$$A_N := [N^\alpha(a - E); N^\alpha(b - E)],$$

$$B_N := \left[\frac{1}{2}N^\alpha(a - E); \frac{1}{2}N^\alpha(b - E) \right].$$

By the observations above, and the change of variable $u = N^\alpha(x - E)$ and $v = N^\alpha(y - E)$ we get

$$\begin{aligned} & \left| \int (\Xi^{-1}\tilde{f})'(x) d\mu_V(x) \right| \\ & \leq \frac{C}{N^\alpha} \iint_{A_N^2} \left| \frac{f(u) - f(v)}{u - v} \right| \left(\frac{\sigma(E + \frac{u}{N^\alpha})}{\sigma(E + \frac{v}{N^\alpha})} + \frac{1}{\sigma(E + \frac{u}{N^\alpha})\sigma(E + \frac{v}{N^\alpha})} \right) dudv. \end{aligned} \quad (2.25)$$

For large enough N , on the set $(u, v) \in (A_N \setminus B_N)^2$, the function $|f(u) - f(v)|$ is always zero, thus the integral on the right above can be divided into integrals over the sets:

$$(A_N \times A_N) \cap (A_N \setminus B_N \times A_N \setminus B_N)^c = B_N \times B_N \cup B_N \times (A_N \setminus B_N) \cup (A_N \setminus B_N) \times B_N. \quad (2.26)$$

We bound the integral in (2.25) over each set in (2.26). We begin with the first set in (2.26). For $(u, v) \in B_N \times B_N$, $\sigma(E + \frac{u}{N^\alpha})$ and $\sigma(E + \frac{v}{N^\alpha})$ are uniformly bounded above and below. Therefore, the integral in (2.25) can be bounded in this region by

$$\begin{aligned} & \iint_{B_N^2} \left| \frac{f(u) - f(v)}{u - v} \right| dudv \\ & = \iint_{[-M; M]^2} \left| \frac{f(u) - f(v)}{u - v} \right| dudv + 2 \int_{-M}^M \int_{B_N \cap \{|u| \geq M\}} \left| \frac{f(v)}{u - v} \right| du dv, \end{aligned}$$

the integral over $[-M; M]^2$ exists by the differentiability of f , while:

$$\int_{-M}^M \int_{B_N \cap \{|u| \geq M\}} \left| \frac{f(v)}{u - v} \right| du dv \leq C \int_{-M}^M |f(v)| \log[N|v + M||v - M|] dv \leq C \log N,$$

for N large enough.

For the second set in (2.25), observe that for $(u, v) \in B_N \times (A_N \setminus B_N)$, $f(v)$ is 0 for N sufficiently large, and $\sigma(E + \frac{u}{N^\alpha})$ is bounded uniformly above and below while $f(u)$ is 0 outside $[-M; M]$. This implies that the integral in (2.25) can be bounded in this region by

$$\begin{aligned} & \int_{A_N \setminus B_N} \int_{-M}^M \left| \frac{f(u)}{u - v} \right| \left(\frac{\sigma(E + \frac{u}{N^\alpha})}{\sigma(E + \frac{v}{N^\alpha})} + \frac{1}{\sigma(E + \frac{u}{N^\alpha})\sigma(E + \frac{v}{N^\alpha})} \right) dudv \\ & \leq \frac{C\|f\|_{C(\mathbb{R})}}{N^\alpha} \int_{A_N \setminus B_N} \frac{1}{\sigma(E + \frac{v}{N^\alpha})} dv \leq C, \end{aligned}$$

where in the final line we used $|u - v| \geq cN^\alpha$ for $u \in [-M; M]$, $v \in A_N \setminus B_N$.

We can do similarly for the third set in (2.25) and putting together these bounds on the right hand side of (2.25) gives

$$\left| \int (\Xi^{-1} \tilde{f})'(x) d\mu_V(x) \right| \leq \frac{C \log N}{N^\alpha},$$

which is $\omega(1)$ as claimed.

We continue with (2.23). Recall that we reduced this problem to computing the limit of $\mu_V(\Xi^{-1}(\tilde{f})\tilde{f}')$. Using the inversion formula we see that

$$\int \Xi^{-1} \tilde{f}(x) \tilde{f}'(x) d\mu_V(x) = -\frac{1}{\beta\pi^2} \int_a^b \int_a^b \frac{\sigma(x) \tilde{f}'(x) (\tilde{f}(x) - \tilde{f}(y))}{\sigma(y)(x - y)} dx dy$$

Observe that

$$\begin{aligned} \frac{1}{2} \partial_x (\tilde{f}(x) - \tilde{f}(y))^2 &= \tilde{f}'(x) (\tilde{f}(x) - \tilde{f}(y)), \\ \partial_x \left(\frac{\sigma(x)}{x - y} \right) &= \frac{-\frac{1}{2}(a + b)(x + y) + ab + xy}{\sigma(x)(x - y)^2}. \end{aligned}$$

Therefore, integration by parts yields

$$\begin{aligned} \int \Xi^{-1} \tilde{f}(x) \tilde{f}'(x) d\mu_V(x) &= -\frac{1}{2\beta\pi^2} \int_a^b \int_a^b \frac{\sigma(x) \partial_x (\tilde{f}(x) - \tilde{f}(y))^2}{\sigma(y)(x - y)} dx dy \\ &= \frac{1}{2\beta\pi^2} \int_a^b \int_a^b \left(\frac{\tilde{f}(x) - \tilde{f}(y)}{x - y} \right)^2 \left(\frac{ab + xy - \frac{1}{2}(a + b)(x + y)}{\sigma(x)\sigma(y)} \right) dx dy, \end{aligned}$$

By changing variables again to $(u, v) = (N^\alpha(x - E), N^\alpha(y - E))$ and observing that

$$ab + xy - \frac{1}{2}(a + b)(x + y) = -\sigma(E)^2 + \frac{u + v}{N^\alpha} \left(\frac{a + b}{2} + E \right) + \frac{uv}{N^{2\alpha}},$$

we obtain

$$\begin{aligned} \int \Xi^{-1} \tilde{f}(x) \tilde{f}'(x) d\mu_V(x) &= -\frac{1}{2\beta\pi^2} \iint_{A_N^2} \left(\frac{f(u) - f(v)}{u - v} \right)^2 \left(\frac{\sigma(E)^2 - \frac{u+v}{N^\alpha} \left(\frac{a+b}{2} + E \right) - \frac{uv}{N^{2\alpha}}}{\sigma(E + \frac{u}{N^\alpha}) \sigma(E + \frac{v}{N^\alpha})} \right) du dv. \quad (2.27) \end{aligned}$$

As before, $(f(u) - f(v))^2$ is zero for all $(u, v) \in (A_N \setminus B_N)^2$ for large enough N , therefore we split the above integral into the regions defined in (2.26).

Notice that uniformly in $u \in B_N$

$$\frac{1}{\sigma(E + \frac{u}{N^\alpha})} = \frac{1}{\sigma(E)} + O\left(\frac{|u|}{N^\alpha}\right),$$

and further notice $(u + v)/N^\alpha$ and $uv/N^{2\alpha}$ are bounded uniformly by constants in the entire region $A_N \times A_N$ and converge pointwise to 0 for each (u, v) .

Consequently the integral (2.27) over the region $B_N \times B_N$ is:

$$\begin{aligned} & \iint_{B_N^2} \left(\frac{f(u) - f(v)}{u - v} \right)^2 \left(\frac{\sigma(E)^2 - \frac{u+v}{N^\alpha} \left(\frac{a+b}{2} + E \right) - \frac{uv}{N^{2\alpha}}}{\sigma(E + \frac{u}{N^\alpha}) \sigma(E + \frac{v}{N^\alpha})} \right) dudv \\ &= \iint_{B_N^2} \left(\frac{f(u) - f(v)}{u - v} \right)^2 \left(1 - \frac{u+v}{N^\alpha \sigma(E)^2} \left(\frac{a+b}{2} + E \right) - \frac{uv}{N^{2\alpha} \sigma(E)^2} \right) dudv \\ & \quad + O \left(\frac{1}{N^\alpha} \iint_{B_N^2} \left(\frac{f(u) - f(v)}{u - v} \right)^2 (|u| + |v|) dudv \right), \quad (2.28) \end{aligned}$$

the first term of (2.28) is equal to,

$$\frac{1}{2\beta\pi^2} \iint \left(\frac{f(u) - f(v)}{u - v} \right)^2 dudv + O\left(\frac{1}{N^\alpha}\right)$$

while the second term in (2.28) can be written as

$$\begin{aligned} \iint_{B_N^2} \left(\frac{f(u) - f(v)}{u - v} \right)^2 (|u| + |v|) dudv &= \iint_{[-M; M]^2} \left(\frac{f(u) - f(v)}{u - v} \right)^2 (|u| + |v|) dudv \\ & \quad + 2 \int_{-M}^M \int_{B_N \cap \{|u| \geq M\}} \left(\frac{f(v)}{u - v} \right)^2 (|u| + |v|) dudv, \end{aligned}$$

the integral over $[-M; M]^2$ is finite by differentiability of f while the second is bounded by

$$\begin{aligned} & \int_{-M}^M \int_{B_N \cap \{|u| \geq M\}} |f(v)|^2 \left(\frac{1}{|u - v|} + \frac{2|v|}{|u - v|^2} \right) dudv \\ & \leq C \int_{-M}^M |f(v)|^2 \left(\frac{1}{|v - M|} + \frac{1}{|M + v|} + \log[N|v - M||v + M|] \right) dv \leq C \log N \end{aligned}$$

since $\text{supp } f \subset [-M, M]$.

In the region $(u, v) \in B_N \times (A_N \setminus B_N)$, $\sigma(E + \frac{u}{N^\alpha})$ is bounded above and below while, for N large enough $f(v) = 0$, thus the integral over $B_N \times (A_N \setminus B_N)$ is bounded above by

$$\begin{aligned} & \int_{A_N \setminus B_N} \int_{B_N} \left(\frac{f(u) - f(v)}{u - v} \right)^2 \left(\frac{1}{\sigma(E + \frac{u}{N^\alpha}) \sigma(E + \frac{v}{N^\alpha})} \right) dudv \\ & \leq \int_{A_N \setminus B_N} \int_{-M}^M \left(\frac{f(u)}{u - v} \right)^2 \frac{1}{\sigma(E + \frac{v}{N^\alpha})} dudv \leq \frac{C}{N^{2\alpha}} \int_{A_N \setminus B_N} \frac{1}{\sigma(E + \frac{v}{N^\alpha})} dv \leq \frac{C}{N^\alpha}, \end{aligned}$$

where in the second line we used $|u - v| \geq cN^\alpha$ for $u \in [-M; M]$ and $v \in A_N \setminus B_N$. By symmetry of the integrand in (2.27) this argument extends to the region $(u, v) \in (A_N \setminus B_N) \times B_N$.

Altogether, our bounds show

$$\int \Xi^{-1} \tilde{f}(x) \tilde{f}'(x) d\mu_V(x) = -\frac{1}{2\beta\pi^2} \iint \left(\frac{f(x) - f(y)}{x - y} \right)^2 dx dy + O\left(\frac{\log N}{N^\alpha}\right),$$

which shows (2.23).

We conclude by proving (2.24). The proof will be similar to the proof of 2.7. As in 2.7 we may restrict our attention to the event $\Omega = \{\forall i : |\lambda_i - \gamma_i| \leq N^{-\frac{2}{3} + \xi_i^{\frac{1}{3}}}\}$ by applying 2.5. Further, we use again the sets J_1 , J_2 , and J_3 defined in 2.6.

Define for $j \in \{1, 2, 3\}$:

$$M_N^{(j)} = \sum_{i \in J_j} (\delta_{\lambda_i} - N \mathbb{1}_{[\gamma_i, \gamma_{i+1}]} \mu_V)$$

so that $M_N = M_N^{(1)} + M_N^{(2)} + M_N^{(3)}$. We can write

$$\begin{aligned} \iint \frac{\Xi^{-1}(\tilde{f})(x) - \Xi^{-1}(\tilde{f})(y)}{x - y} dM_N(x) dM_N(y) \\ = \sum_{1 \leq j_1, j_2 \leq 3} \iint \frac{\Xi^{-1}(\tilde{f})(x) - \Xi^{-1}(\tilde{f})(y)}{x - y} dM_N^{(j_1)}(x) dM_N^{(j_2)}(y) \end{aligned}$$

Integrating repeatedly for each (j_1, j_2) yields:

$$\begin{aligned} \iint \frac{\Xi^{-1}(\tilde{f})(x) - \Xi^{-1}(\tilde{f})(y)}{x - y} dM_N^{(j_1)}(x) dM_N^{(j_2)}(y) = \\ N^2 \sum_{\substack{i_1 \in J_{j_1} \\ i_2 \in J_{j_2}}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} d\mu_V(x_1) \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} d\mu_V(x_2) \int_T du dv dt \left\{ (\lambda_{i_1} - x_1)(\lambda_{i_2} - x_2)t(1 - t) \right. \\ \left. \times \Xi^{-1}(\tilde{f})^{(3)}(tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + u(\lambda_{i_2} - x_2) + t(x_1 - x_2) + x_2) \right\} \end{aligned} \quad (2.29)$$

where $T = [0; 1]^3$. We will bound (2.29) for each pair (j_1, j_2) .

For $(j_1, j_2) = (1, 1)$. Recall by 2.6 (c) that $|J_1| \leq CN^{1-\alpha}$, and further from the proof of 2.6 uniformly in $i \in J_1$, $|\lambda_i - x| \leq CN^{\xi-1}$ whenever $x \in [\gamma_i, \gamma_{i+1}]$. We use (2.29), 2.4 eq. (2.8) to obtain the upper bound

$$\begin{aligned} \iint \frac{\Xi^{-1}(\tilde{f})(x) - \Xi^{-1}(\tilde{f})(y)}{x - y} dM_N^{(1)}(x) dM_N^{(1)}(y) \leq \\ N^2 \sum_{\substack{i_1 \in J_1 \\ i_2 \in J_1}} \int_{\gamma_{i_1}}^{\gamma_{i_1+1}} \int_{\gamma_{i_2}}^{\gamma_{i_2+1}} N^{3\alpha} \log N |\lambda_{i_1} - x_1| |\lambda_{i_2} - x_2| d\mu_V(x_1) d\mu_V(x_2) \leq CN^{2\xi+\alpha} \log N, \end{aligned}$$

which is $\omega(1)$ when divided by N .

For $(j_1, j_2) = (2, 2)$. We remark that the strategy is not as straightforward as the case $i \in J_2$ in the proof of 2.7 eq. (2.18), this is because the term $t(x_1 - x_2) + x_2$ appearing as an argument in (2.29) may enter a neighborhood of 0 depending on the indices $i_1, i_2 \in J_2$; so we may not use the bound 2.4 eq. (2.9) uniformly in $i_1, i_2 \in J_2$. Some care is needed also because M_N is a signed measure so $|M_N(g)|$ need not be bounded by $M_N(|g|)$.

It will be convenient to use directly eq. (2.11) from the proof of 2.4 (this can be done as J_2 is located outside the support of f). We can write

$$\begin{aligned}
& \frac{\Xi^{-1}(\tilde{f})(x) - \Xi^{-1}(\tilde{f})(y)}{x - y} \\
&= \frac{1}{\beta\pi^2} \int_{-M}^M \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})(x - y)} \left(\frac{1}{S(y)(u - N^\alpha(y - E))} - \frac{1}{S(x)(u - N^\alpha(x - E))} \right) du \\
&= \frac{1}{\beta\pi^2} \int_{-M}^M \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})} \left\{ \frac{S(x) - S(y)}{(x - y)} \frac{1}{S(x)S(y)(u - N^\alpha(y - E))} \right. \\
&\quad \left. + \frac{N^\alpha}{S(x)(u - N^\alpha(x - E))(u - N^\alpha(y - E))} \right\} du. \quad (2.30)
\end{aligned}$$

When we integrate the term on the third line of (2.30) against $M_N^{(2)} \otimes M_N^{(2)}$, we obtain

$$\int_{-M}^M \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})} \left\{ \int M_N^{(2)} \left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)} dt \right) \frac{1}{(u - N^\alpha(y - E))} dM_N^{(2)}(y) \right\} du, \quad (2.31)$$

define the function

$$g(y) := M_N^{(2)} \left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)} dt \right),$$

first, $g(y)$ is bounded for any $y \in [a; b]$:

$$\begin{aligned}
& \left| M_N^{(2)} \left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)} dt \right) \right| \\
&= \left| \frac{N}{S(y)} \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 \left(\frac{S'(t(\lambda_i - y) + y)}{S(\lambda_i)} - \frac{S'(t(x - y) + y)}{S(x)} \right) dt d\mu_V(x) \right| \\
&\leq \left| \frac{N}{S(y)} \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 \frac{S'(t(\lambda_i - y) + y) - S'(t(x - y) + y)}{S(\lambda_i)} dt d\mu_V(x) \right| \\
&\quad + \left| \frac{N}{S(y)} \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \int_0^1 \frac{S(x) - S(\lambda_i)}{S(x)S(\lambda_i)} S'(t(x - y) + y) dt d\mu_V(x) \right| \leq CN^\xi,
\end{aligned}$$

where in the final line we used S and S' are smooth on $[a; b]$ (and therefore uniformly Lipschitz), $S > 0$ in a neighborhood of $[a; b]$, further $|x - \lambda_i| \leq CN^{\xi-1}$, and $|J_2| \leq CN$. Moreover, $g(y)$ is uniformly Lipschitz in $[a; b]$ with constant CN^ξ , since:

$$\begin{aligned}
& M_N^{(2)} \left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)} - \frac{S'(t(\cdot - z) + z)}{S(\cdot)S(z)} dt \right) = \\
& (z - y) M_N^{(2)} \left(\int_0^1 \int_0^1 \frac{tS''(ut(z - y) + t(\cdot - z) + y)}{S(\cdot)S(y)} dt du \right) \\
& \quad + \frac{S(z) - S(y)}{S(z)S(y)} M_N^{(2)} \left(\int_0^1 \frac{S'(t(\cdot - z) + z)}{S(\cdot)} dt \right)
\end{aligned}$$

and both terms appearing in $M_N^{(2)}$ above are of the same form as g so they are bounded by CN^ξ . Returning to (2.31), we may bound

$$\begin{aligned} & \left| M_N^{(2)} \left(\frac{g(y)}{u - N^\alpha(y - E)} \right) \right| \\ &= \left| N \sum_{i \in J_2} \int_{\gamma_i}^{\gamma_{i+1}} \frac{g(\lambda_i) - g(x)}{(u - N^\alpha(\lambda_i - E))} + \frac{N^\alpha(\lambda_i - x)g(x)}{(u - N^\alpha(\lambda_i - E))(u - N^\alpha(x - E))} d\mu_V(x) \right| \\ &\leq \int_{[a; b] \cap \{|x - E| \geq \frac{2M}{N^\alpha}\}} \frac{CN^{2\xi}}{|u - N^\alpha(x - E)|} + \frac{CN^{2\xi + \alpha}}{(u - N^\alpha(x - E))^2} dx \\ &\leq CN^{2\xi - \alpha} \log N + CN^{2\xi}, \end{aligned}$$

uniformly in u . Thus (2.31) is bounded by $CN^{2\xi}$ as f is bounded.

The remaining term in (2.30) is

$$N^\alpha \int_{-M}^M \frac{f(u)}{\sigma(E + \frac{u}{N^\alpha})} M_N^{(2)} \left(\frac{1}{S(\cdot)(u - N^\alpha(\cdot - E))} \right) M_N^{(2)} \left(\frac{1}{u - N^\alpha(\cdot - E)} \right) du. \quad (2.32)$$

Repeating our argument in the previous paragraph gives:

$$\begin{aligned} \left| M_N^{(2)} \left(\frac{1}{S(\cdot)(u - N^\alpha(\cdot - E))} \right) \right| &\leq CN^{\xi - \alpha} \log N + CN^\xi, \\ \left| M_N^{(2)} \left(\frac{1}{u - N^\alpha(\cdot - E)} \right) \right| &\leq CN^\xi, \end{aligned}$$

where in the first inequality we use $1/S$ is uniformly bounded and uniformly Lipschitz on $[a; b]$. Inserting the bounds into (2.32) gives an upper bound of $CN^{2\xi + \alpha}$, as f is bounded.

Altogether

$$\left| \frac{\Xi^{-1}(\tilde{f})(x) - \Xi^{-1}(\tilde{f})(y)}{x - y} dM_N^{(2)}(x) dM_N^{(2)}(y) \right| \leq CN^{2\xi},$$

which is $\omega(1)$ when divided by N .

For $(j_1, j_2) = (3, 3)$. We bound as in the previous case, except now we define

$$g(y) = M_N^{(3)} \left(\int_0^1 \frac{S'(t(\cdot - y) + y)}{S(\cdot)S(y)} dt \right),$$

and apply (for x in the region defined in J_3):

$$|\lambda_i - x| \leq N^{-\frac{2}{3} + \xi} \hat{i}^{-\frac{1}{3}}, \quad \left| \frac{1}{(u - N^\alpha(x - E))} \right| \leq \frac{C}{N^\alpha},$$

to obtain

$$\begin{aligned} \left| M_N^{(3)} \left(\frac{g(y)}{u - N^\alpha(y - E)} \right) \right| &\leq CN^{2\xi - \alpha}, \\ \left| M_N^{(3)} \left(\frac{1}{S(\cdot)(u - N^\alpha(\cdot - E))} \right) \right| &\leq CN^{\xi - \alpha}, \\ \left| M_N^{(3)} \left(\frac{1}{u - N^\alpha(\cdot - E)} \right) \right| &\leq CN^{\xi - \alpha}, \end{aligned}$$

altogether giving

$$\left| \iint \frac{\Xi^{-1}(\tilde{f})(x) - \Xi^{-1}(\tilde{f})(y)}{x - y} dM_N^{(3)}(x) dM_N^{(3)}(y) \right| \leq CN^{2\xi-\alpha},$$

which is $\omega(1)$ when divided by N .

For $(j_1, j_2) = (1, 2)$. By the bounds $|\lambda_{i_j} - \gamma_{i_j}| \leq CN^{\xi-1}$, $|\gamma_{i_j} - x_j| \leq \frac{C}{N}$ for $x_j \in [\gamma_{i_j}; \gamma_{i_j+1}]$, whenever

$$N^\alpha |tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + t(x_1 - x_2) + u(\lambda_{i_2} - x_2) + x_2 - E| \geq M + 1, \quad (2.33)$$

we have

$$|t(\gamma_{j_1} - \gamma_{j_2}) + (\gamma_{j_2} - E)| + CN^{\xi-1} \geq M + 1,$$

by triangle inequality. It follows that for N sufficiently large, uniformly in $x_1 \in [\gamma_{i_1}; \gamma_{i_1+1}]$, $x_2 \in [\gamma_{i_2}; \gamma_{i_2+1}]$, $u, v \in [0; 1]$

$$\begin{aligned} & \frac{1}{|tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + t(x_1 - x_2) + u(\lambda_{i_2} - x_2) + x_2 - E|} \\ & \leq \frac{C}{|t(\gamma_{i_1} - \gamma_{i_2}) + (\gamma_{i_2} - E)|}, \end{aligned}$$

where the constant C only depends on M . Therefore, whenever (2.33) is satisfied, applying 2.4 eq. (2.9) yields

$$\begin{aligned} & \left| \Xi^{-1}(\tilde{f})^{(3)}(tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + t(x_1 - x_2) + u(\lambda_{i_2} - x_2) + x_2) \right| \\ & \leq \frac{C}{N^\alpha ((t(\gamma_{i_1} - \gamma_{i_2}) + \gamma_{i_2} - E)^4 \wedge 1)}. \quad (2.34) \end{aligned}$$

Now, let $t \in (0, 1)$ fixed and define the sets

$$\begin{aligned} K_t^1 &:= \left\{ j \in J_2, \ t \left(E - \frac{2M}{N^\alpha} - \gamma_j \right) + \gamma_j - E \geq \frac{2M}{N^\alpha} \right\}, \\ K_t^2 &:= \left\{ j \in J_2, \ t \left(E + \frac{2M}{N^\alpha} - \gamma_j \right) + \gamma_j - E \leq -\frac{2M}{N^\alpha} \right\}, \\ K_t &:= K_t^1 \cup K_t^2. \end{aligned}$$

By construction, if $i_2 \in K_t^1$ then

$$|t(\gamma_{i_1} - \gamma_{i_2}) + (\gamma_{i_2} - E)| \geq \frac{2M}{N^\alpha}$$

uniformly in $i_1 \in J_1$. Thus for such $i_2 \in K_t^1$, (2.33) is satisfied for N sufficiently large (uniformly in u, v, x_1 , and x_2 ; also the choice of how large N must be only depends on ξ and μ_V). The same statement holds for K_t^2 .

We now proceed to bound (2.29) for $j_1 = 1$ and $j_2 = 2$ by splitting J_2 into the regions K_t^1 , K_t^2 and $J_2 \setminus K_t$. We start with K_t^1 (the argument for K_t^2 is identical). Our observations

from the previous paragraph along with (2.34) gives:

$$\begin{aligned}
& \int_T du dv dt \left| N^2 \sum_{\substack{i_1 \in J_1 \\ i_2 \in K_t^1}} \int_{\gamma_{i_1}}^{\gamma_{i_1}+1} d\mu_V(x_1) \int_{\gamma_{i_2}}^{\gamma_{i_2}+1} d\mu_V(x_2) \left\{ (\lambda_{i_1} - x_1)(\lambda_{i_2} - x_2)t(1-t) \right. \right. \\
& \quad \left. \left. \times \Xi^{-1}(\tilde{f})^{(3)}(tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + u(\lambda_{i_2} - x_2) + t(x_1 - x_2) + x_2) \right\} \right| \\
& \leq \int_0^1 \sum_{\substack{i_1 \in J_1 \\ i_2 \in K_t^1}} \frac{CN^{2\xi-2-\alpha}t(1-t)}{(t(\gamma_{i_1} - \gamma_{i_2}) + (\gamma_{i_2} - E))^4} dt \leq \int_0^1 \sum_{i_2 \in K_t^1} \frac{CN^{2\xi-1-2\alpha}t(1-t)}{((1-t)(\gamma_{i_2} - E) - \frac{t2M}{N^\alpha})^4} dt
\end{aligned}$$

where in the final line we used $|J_1| \leq CN^{1-\alpha}$ from 2.6 (c). Next, note that

$$\begin{aligned}
& \frac{1}{N} \sum_{i_2 \in K_t^1} \frac{1}{((1-t)(\gamma_{i_2} - E) - \frac{t2M}{N^\alpha})^4} \\
& \leq C \int_{E + \frac{2M}{N^\alpha}(\frac{1+t}{1-t})}^{E + \frac{1}{2}(E-a) \wedge (b-E)} \frac{dx}{((1-t)(x - E) - \frac{t2M}{N^\alpha})^4} \leq \frac{CN^{3\alpha}}{1-t},
\end{aligned}$$

since, by definition of K_t^1 , $\gamma_{i_2} \geq E + \frac{2M}{N^\alpha} \left(\frac{1+t}{1-t} \right)$. We conclude,

$$\int_0^1 \sum_{i_2 \in K_t^1} \frac{CN^{2\xi-1-2\alpha}t(1-t)}{((1-t)(\gamma_{i_2} - E) - \frac{t2M}{N^\alpha})^4} dt \leq CN^{2\xi+\alpha}.$$

We continue with $J_2 \setminus K_t$. By the same argument as in 2.6 (c) $|J_2 \setminus K_t| \leq \frac{CN^{1-\alpha}}{1-t}$ where the constant C does not depend on t , we use this in addition with 2.4 eq. (2.9), $|J_1| \leq CN^{1-\alpha}$, and $|\lambda_{i_j} - x_j| \leq CN^{\xi-1}$ to obtain the bound

$$\begin{aligned}
& \int_T du dv dt \left| N^2 \sum_{\substack{i_1 \in J_1 \\ i_2 \in J_2 \setminus K_t}} \int_{\gamma_{i_1}}^{\gamma_{i_1}+1} d\mu_V(x_1) \int_{\gamma_{i_2}}^{\gamma_{i_2}+1} d\mu_V(x_2) \left\{ (\lambda_{i_1} - x_1)(\lambda_{i_2} - x_2)t(1-t) \right. \right. \\
& \quad \left. \left. \times \Xi^{-1}(\tilde{f})^{(3)}(tv(\lambda_{i_1} - x_1) + ut(x_2 - \lambda_{i_2}) + u(\lambda_{i_2} - x_2) + t(x_1 - x_2) + x_2) \right\} \right| \\
& \leq C \int_0^1 N^{3\alpha} \log N \times N^{2\xi-2} \times N^{2-2\alpha}t dt \leq CN^{\alpha+2\xi} \log N.
\end{aligned}$$

Combining the bounds we have obtained gives

$$\left| \iint \frac{\Xi^{-1}\tilde{f}(x) - \Xi^{-1}\tilde{f}(y)}{x-y} dM_N^{(1)}(x) dM_N^{(2)}(y) \right| \leq CN^{\alpha+2\xi} \log N,$$

which is $\omega(1)$ when divided by N for ξ small enough.

For $j_1 = 1$ or 2 and $j_2 = 3$. the proof is similar and we omit the details. \square

2.3 Proof of 1.4

We proceed with the proof of 1.4. Applying the loop equation (2.5) to $h = \Xi^{-1}(\tilde{f})$ yields

$$F_1^N(\Xi^{-1}(\tilde{f})) = M_N(\tilde{f}) + \left(1 - \frac{\beta}{2}\right) L_N((\Xi^{-1}\tilde{f})') + \frac{1}{N} \left[\frac{\beta}{2} \iint \frac{\Xi^{-1}\tilde{f}(x) - \Xi^{-1}\tilde{f}(y)}{x - y} dM_N(x) dM_N(y) \right].$$

Combining 2.8 eq. (2.22) and eq. (2.24) we get

$$F_1^N(\Xi^{-1}(\tilde{f})) = M_N(\tilde{f}) + \omega(1). \quad (2.35)$$

Using the first loop equation from 2.1, and the fact that $M_N(\tilde{f})$ is bounded by $2N\|f\|_{C(\mathbb{R})}$ gives

$$\mathbb{E}_V^N(M_N(\tilde{f})) = o(1). \quad (2.36)$$

We now show recursively that

$$F_k^N(\Xi^{-1}(\tilde{f}), \tilde{f}, \dots, \tilde{f}) = \tilde{M}_N(\tilde{f})^k - (k-1)\sigma_f^2 \tilde{M}_N(\tilde{f})^{k-2} + \omega(1). \quad (2.37)$$

Here, the set on which the bound holds might vary from one k to another but each bound has probability greater than $1 - e^{-N^{c_k}}$ for each fixed k .

The bound holds for $k = 1$, by (2.35). Now, assume this holds for $k \geq 1$. Then by Proposition 2.1 we have

$$F_{k+1}^N(\Xi^{-1}(\tilde{f}), \tilde{f}, \dots, \tilde{f}) = F_k^N(\Xi^{-1}(\tilde{f}), \tilde{f}, \dots, \tilde{f}) \tilde{M}_N(\tilde{f}) + \tilde{M}_N(\tilde{f})^{k-1} L_N(\Xi^{-1}(\tilde{f})\tilde{f}') \quad (2.38)$$

On a set of probability greater than $1 - e^{-N^{c_{k+1}}}$ we have by the induction hypothesis, 2.7 eq. (2.17), and 2.8 eq. (2.23), for some $\delta > 0$ and a constant C

$$|F_k^N(\Xi^{-1}(\tilde{f}), \tilde{f}, \dots, \tilde{f}) - \tilde{M}_N(\tilde{f})^k - (k-1)\sigma_f^2 \tilde{M}_N(\tilde{f})^{k-2}| \leq \frac{C}{N^\delta},$$

$$|L_N(\Xi^{-1}(\tilde{f})\tilde{f}') + \sigma_f^2| \leq \frac{C}{N^\delta},$$

$$|M_N(\tilde{f})| \leq N^{\delta/2k}.$$

And this proves the induction. Using the fact that F_k is bounded polynomially and deterministically, the computation of the moments is then straightforward and this concludes the proof of Theorem 1.4.

Remark 2.9. *The same proof would also show the macroscopic central limit Theorem already shown in [4, 18, 14] but with less restrictive condition $V \in C^6(\mathbb{R})$ and $f \in C^5(\mathbb{R})$ with appropriate decay conditions.*

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